

3. Nikol'skii, M. S., A direct method in linear differential games with general integral constraints. *Differentsial'nye Uravneniia*, Vol. 8, №6, 1972.
4. Pozharitskii, G. K., Impulse tracking of a point with bounded thrust. *PMM Vol. 37, №2, 1973.*
5. Ukhobotov, V. I., On linear differential games with impulse controls on a finite time interval. *Proc. First Conf. Young Scientists Faculty Comput. Math. and Cybernetics*, Moscow, Izd. Mosk. Gos. Univ., 1973.
6. Krasovskii, N. N. and Subbotin, A. I., An alternative for the game problem of convergence. *PMM Vol. 34, №6, 1970.*
7. Krasovskii, N. N. and Subbotin, A. I., Approximation in a differential game. *PMM Vol. 37, №2, 1973.*
8. Krasovskii, N. N., A differential game of encounter-evasion, I, II. *Izv. Akad. Nauk SSSR, Tekhn. Kibernetika*, №№2, 3, 1973.
9. Halanay, A. and Wexler, D., *Qualitative Theory of Sampled-Data Systems*, Moscow, "Mir", 1971.

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NEW PERIODIC SOLUTIONS FOR THE PROBLEM OF MOTION OF A HEAVY SOLID BODY AROUND A FIXED POINT

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The theory of generation of periodic solutions in canonic systems of near-integrable differential equations was developed by Poincaré for the purposes of celestial mechanics. In this paper we establish the applicability of these results to the classical problem of the motion of a heavy solid body with a fixed point. By the same token we have succeeded in essentially widening the class of periodic solutions appearing in this problem.

1. Perturbation of uniform rotations. The Hamiltonian function of the problem being analyzed has the form

$$F = F_0 + \mu F_1 \quad (1.1)$$

Here F_0 is the kinetic energy, μF_1 is the system's potential energy (the chosen constant multiplier μ is the product of the body's weight by the distance from the center of gravity to the point of fixing). Canonic equations with Hamiltonian (1.1) have a cyclic integral, i.e., an area integral; by fixing it constant, we reduce the problem being examined to a system with two degrees of freedom, which we call reduced problem. When $\mu = 0$, we have the Euler-Poinsot case. In this unperturbed problem there exist particular isolated periodic solutions, namely, uniform rotations around the principal axes of the inertia ellipsoid. Let us ascertain whether the equations with Hamiltonian function (1.1) admit of periodic solutions if $\mu \neq 0$ but is very small.

Case of a nonsymmetric solid body.

Theorem 1. Periodic solutions of the unperturbed reduced problem — nonvertical constant rotations around the principal inertia axes — do not vanish under the addition of a perturbation, but for small μ turn into periodic solutions of the perturbed problem depending analytically on the small parameter μ and on the energy constant.

Consequently, at almost all three-dimensional energy levels the perturbed reduced problem has six periodic solutions for small values of μ .

Since we are examining the nonvertical uniform rotations of the unperturbed problem to prove this assertion we can pass to the canonic Deprit variables (l, g, h, L, G, H) [1]. Function (1.1) written in these coordinates do not contain the angular variable h . Therefore, H is the first integral of the canonic equations (the area integral); a lowering of the system's order is achieved by fixing it constant. Let A, B, C be the principal central moments of inertia of the solid. We assume $A > B > C$. The Hamiltonian of the Euler-Poinsot problem in the Deprit variables has the form

$$F_0 = \frac{1}{2} \left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B} \right) (G^2 - L^2) + \frac{L^2}{2C}$$

Let us consider the rotation around the minor inertia axis (with period T)

$$L = 0, \quad G = G_0, \quad l = \frac{\pi}{2}, \quad g = \frac{G_0}{A} t + g_0, \quad T = \frac{2\pi A}{G_0} \quad (1.2)$$

Applying the small parameter method, we denote the deviations from the periodic solution (1.2) when $\mu = 0$ by L_1, G_1, l_1 and g_1 , respectively. Then the linear equations

$$L_1^* = \frac{B-A}{AB} G_0^2 l_1, \quad G_1^* = 0, \quad l_1^* = \frac{A-C}{AC} L_1, \quad g_1^* = \frac{G_1}{A} \quad (1.3)$$

are the equations in variations for the generating solutions. They are easily integrated

$$G_1 = G_{1,0}, \quad g_1 = \frac{G_{1,0}}{A} t + g_{1,0}, \quad L_1 = A_1 \sin \omega t + B_1 \cos \omega t$$

$$l_1 = A_2 \sin \omega t + B_2 \cos \omega t, \quad \omega^2 = \frac{(A-B)(A-C)}{BC} \left(\frac{G_0}{A} \right)^2 > 0$$

$$B_1 = L_{1,0}, \quad B_2 = l_{1,0}, \quad A_1 = \frac{B-A}{AB} \frac{G_0^2}{\omega} l_{1,0}, \quad A_2 = \frac{A-C}{AC} \frac{L_{1,0}}{\omega}$$

We note that

$$\frac{A_1 A_2}{l_{1,0} L_{1,0}} = \frac{(B-A)(A-C)}{A^2 BC} \left(\frac{G_0}{\omega} \right)^2 = -1$$

The monodromy matrix of Eqs. (1.3) is

$$X(T) = \begin{pmatrix} \cos(\omega T) & 0 & \frac{A_1}{l_{1,0}} \sin(\omega T) & 0 \\ 0 & 1 & 0 & 0 \\ \frac{A_2}{L_{1,0}} \sin(\omega T) & 0 & \cos(\omega T) & 0 \\ 0 & \frac{2\pi}{G_0} & 0 & 1 \end{pmatrix}$$

According to the modification of Poincaré's small parameter method, proposed in [2], we need to form the matrix $Y(T) = X(T) - E$ and ascertain that the rank of this

matrix equals three. It can be shown that this condition is satisfied; consequently, in the system with Hamiltonian (1.1) there exist periodic solutions depending analytically on μ , whose period differs slightly from T .

By crossing out the last column and the second row from matrix $Y(T)$ we obtain a matrix whose determinant is

$$4\pi G_0^{-1}[\cos(\omega T) - 1]$$

In order that the determinant be nonzero, it is necessary to require the fulfillment of the condition $\omega T \neq 2\pi k$ (k is an integer) or, equivalently,

$$\{(A - B)(A - C) / (BC)\}^{1/2} \neq k$$

Let us show that this inequality is always valid. In fact, otherwise

$$A = B + C + (k^2 - 1)BC / A$$

For $k \neq 0$ the last relation contradicts the triangle inequality $A < B + C$, while for $k = 0$ it follows easily from the condition $A > B > C$. Thus, the rank of matrix Y equals three.

Let us establish that the periodic solutions of the perturbed problem depend analytically on the energy constant. To do this we set up the following fifth-order matrix

$$Z = \begin{vmatrix} Y & \varphi \\ \psi & 0 \end{vmatrix}$$

where φ is the column vector of the right-hand side of the unperturbed system of equations, while ψ is the row $(\partial F_0 / \partial L, \partial F_0 / \partial G, \partial F_0 / \partial l, \partial F_0 / \partial g)$, into which the solution (1.2) with $t = T$ have been substituted. It can be shown that the rank of matrix Z equals four; therefore [2], the solutions found depend analytically on the constant energy integral. In fact, by crossing out from Z the last column and the second row, we obtain a matrix whose determinant is

$$-(G_0 / A)^2 [\cos(\omega T) - 1]$$

As was shown above this quantity never vanishes.

For uniform rotations around the major and the mean axes of inertia the theorem is proved in exactly the same way (with the sole difference that in the case of the mean axis the solution of Eqs. (1.3) is expressed not in trigonometric but in hyperbolic functions of time).

Case of dynamic symmetry.

Theorem 2. If $A = B \neq C$, then two periodic solutions of the unperturbed reduced problem — nonvertical constant rotations around the axis of symmetry in opposite directions — do not vanish under the addition of a perturbation, but for small μ turn into periodic solutions of the perturbed problem, depending analytically on parameter μ and on the constant energy integral.

The proof of this assertion is analogous to the proof of Theorem 1.

Notes. 1. The case $A = B = C$ is not examined because it relates to the integrable cases.

2. A number of special cases of integrability are known for the problem being analyzed [3]. On the whole they are periodic solutions expressed in finite form in terms of known functions. Some of them (for example, the Bobylev-Steklov solutions) are, for small values of parameter μ , special cases of the periodic solutions whose existence is

proved in Theorems 1 and 2.

2. The generation of isolated periodic solutions from families of periodic solutions of the Euler-Poinsot problem. The proof of Theorems 1 and 2 did not rely on the actual form of the perturbing function but used only its invariance relative to vertical rotations and its analyticity. Let us indicate a set of new periodic solutions whose existence is closely related with the properties of the whole Hamiltonian function of the problem under analysis.

Case of a nonsymmetric solid body. We assume $A > B > C$. In the reduced Euler-Poinsot problem we introduce in the usual manner the variables action-angle (see [4])

$$I_2 = G, \quad I_1(F_0, I_2) = \frac{1}{2\pi} \oint \sqrt{\frac{2F_0 - I_2^2 f(x)}{C^{-1} - f(x)}} dx \quad (2.1)$$

$$f(x) = A^{-1} \sin^2 x + B^{-1} \cos^2 x$$

By φ_1 and φ_2 we denote the canonical coordinates adjoint to I_1 and I_2 . In a real motion the variables L and G are connected by the inequalities $G \geq 0$, $|L| \leq G$; therefore, the region of possible values of I_1 and I_2 is $\Delta = \{I_1, I_2: I_2 \geq 0, |I_1| \leq I_2\}$. In the canonic variables action-angle the Hamiltonian F_0 depends only on I_1 and I_2 , i.e. $F_0 = F_0(I_1, I_2)$. Using formula (2.1) it is easy to get that the level lines of the function $2F_0(I_1, I_2) / I_2^2$ in the "action" coordinates are straight lines passing through the origin. We note that the straight lines $I_1 = 0$, $|I_1| = I_2$ (lying in Δ) correspond to the uniform rotations of the solid around the major and minor axes of inertia, respectively. The points from Δ located on the two straight lines $2F_0 / I_2^2 = 1 / B$ correspond to the rotations around the mean axis.

The following assertions can be proved with the aid of formula (2.1).

Lemma 1. The function $F_0(I_1, I_2)$ is continuous in Δ and analytic in the region

$$\Delta_A = \Delta \setminus (\{I_1 = 0\} \cup \{2F_0 / I_2^2 = 1/B\} \cup \{|I_1| = I_2\})$$

Lemma 2. The Hessian $|\partial^2 F_0 / \partial I_i \partial I_j|$ ($i, j = 1, 2$) preserves sign in each of the four connected subregions of region Δ_A (cf [5]).

Let us consider Poinsot's geometric representation. When the tangent point (pole) makes one full turn on the inertia ellipsoid, the body turns around the constant moment axis by a certain angle $\alpha = \alpha(2F_0 / I_2^2; A, B, C)$. We set $\omega_i = \partial F_0 / \partial I_i$, $\omega_i = \omega_i(I_1, I_2)$, ($i = 1, 2$) (the frequencies of the Euler-Poinsot problem).

Lemma 3. (See [6], Sect. 86)

$$\omega_2 / \omega_1 = (2\pi)^{-1} \alpha(2F_0 / I_2^2; A, B, C)$$

Lemma 4. The function $\alpha(x; A, B, C)$ is analytic on $(1/A, 1/C) \setminus \{1/B\}$ and $\alpha \rightarrow \infty$ if and only if $x \rightarrow 1/B$ (cf [5]).

It is convenient to denote the area integral by I_3 . The expansion of the perturbing function $F_1(I_1, I_2, I_3, \varphi_1, \varphi_2)$ into a double Fourier series in the variables φ_1 and φ_2 has the following form ([6], Sect. 83):

$$F_1 = \sum_l B_{m,1} e^{i(m\varphi_1 + \varphi_2)} + \sum_l B_{m,-1} e^{i(m\varphi_1 - \varphi_2)} + \sum_l B_{m,0} e^{im\varphi_2} \quad (2.2)$$

Here $B_{m,1}$, $B_{m,-1}$, $B_{m,0}$ are functions of I_1, I_2, I_3 , analytic in region Δ_A for fixed I_3 . By x_0, y_0, z_0 we denote the coordinates of the center of gravity of the body in the

principal inertia axes.

Theorem 3. Let $I_2 \neq 0, I_3 \neq |I_3|$. Consider the set of invariant tori of the reduced Euler-Poinsot problem with rotation numbers

$$(2\pi)^{-1}\alpha (2F_0 / I_2^2; A, B, C) = k$$

where k is an integer. If $z_0 = 0$, then k is odd; if $x_0^2 + y_0^2 = 0, k$ is even; finally, if $z_0 \neq 0$ and $x_0^2 + y_0^2 \neq 0, k$ is any integer.

For any nonsymmetric body there exists $N(A, B, C)$ such that for $k > N$ from the family of periodic solutions lying on each of these tori there are generated under perturbation at least two isolated periodic solutions existing for sufficiently small μ and depending analytically on this parameter. One of them is stable in the first approximation, while the other is unstable. Lemma 4 shows that there exist an infinite number of invariant tori of the Euler-Poinsot problem, with rational rotation numbers, on which at least two isolated periodic solutions of the perturbed problem are generated.

Proof. Suppose that the frequencies ω_1, ω_2 of the unperturbed Euler-Poinsot problem are commensurable for $I = (I_1, I_2) = I^\circ \in \Delta_A$. Then the function $F_1(I^\circ, \omega_1 t, \omega_2 t + \lambda)$ is periodic in t . We denote its time average by $\bar{F}_1(I^\circ, \lambda)$. In order for pairs of isolated periodic solutions to be generated on the torus $I = I^\circ$, it is sufficient to verify the fulfillment of the following conditions ([6], Sects. 42, 79): (1) the Hessian $|\partial^2 F_0 / \partial I^2| \neq 0$ for $I = I^\circ$, (2) $\partial^2 \bar{F}_1 / \partial \lambda^2 \neq 0$ when $\partial \bar{F}_1 / \partial \lambda = 0$ ($I = I^\circ$), (3) the quadratic form

$$\omega_1^2 \frac{\partial^2 F_0}{\partial I_2^2} - 2\omega_1\omega_2 \frac{\partial^2 F_0}{\partial I_1 \partial I_2} + \omega_2^2 \frac{\partial^2 F_0}{\partial I_1^2} \neq 0$$

when $I = I^\circ$.

Condition 1 is satisfied in the whole region Δ_A (Lemma 2). Condition 3 means geometrically that the level line of function $F_0(I_1, I_2)$ does not have an inflection at the point $(I_1, I_2) = I^\circ$. Using formula (2.1) it can be proved that this condition is satisfied everywhere in Δ_A . Expansion (2.2) can be written more precisely (see [4]):

$$F_1 = (\sin \delta \sin \varphi_2, \sin \delta \cos \varphi_2, \cos \delta) (s_{ij}) \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{2.3}$$

$$\cos \delta = \frac{I_3}{I_2}, \quad \alpha = \frac{x_0}{r}, \quad \beta = \frac{y_0}{r}, \quad \gamma = \frac{z_0}{r}, \quad r = \sqrt{x_0^2 + y_0^2 + z_0^2}$$

Here the elements of the square third-order matrix (s_{ij}) are independent of φ_2 ; the expansions of s_{ij} into Fourier series in φ_1 are written out in [4].

We restrict ourselves to the case when $x_0^2 + y_0^2 = 0$, while $z_0 \neq 0$ (the other cases mentioned in the theorem are analyzed analogously). Then, according to (2.3) and [4], we can write F_1 in the following form:

$$F_1 = \gamma (s_{13} \sin \delta \sin \varphi_2 + s_{23} \sin \delta \cos \varphi_2 + s_{33} \cos \delta) = \\ \gamma \frac{2\pi}{K} \frac{\kappa}{\sqrt{\kappa^2 + \Lambda^2}} \left\{ \sin \delta \sum_{n=1}^{\infty} \left[\frac{-q^n (1 - q^{2n}) \operatorname{ch} \sigma \sin 2n \varphi_1 \sin \varphi_2}{1 - 2q^{2n} \operatorname{ch} 2\sigma + q^{4n}} + \right. \right. \\ \left. \left. \frac{q^n (1 + q^{2n}) \operatorname{sh} \sigma \cos 2n \varphi_1 \cos \varphi_2}{1 - 2q^{2n} \operatorname{ch} 2\sigma + q^{4n}} - \frac{1}{4 \operatorname{sh} \sigma} \right] + \right.$$

$$\cos \delta \left(\frac{1}{4} + \sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}} \cos 2n\varphi_1 \right)$$

$$\kappa^2 = \frac{C(A-B)}{A(B-C)}, \quad \Lambda^2 = \kappa^2 \frac{2CF_0 - I_2^2}{I_2^2 - 2AF_0}, \quad q = \exp\left(\frac{-\pi K'}{K}\right)$$

$$K' = K(\sqrt{1-\Lambda^2}), \quad \sigma = \frac{\pi}{2K} F\left(\operatorname{arctg} \frac{\kappa}{\Lambda}, \sqrt{1-\Lambda^2}\right)$$

Here $K(\Lambda)$ is the complete elliptic integral of the first kind with modulus Λ , while F is an elliptic integral of the first kind. We set $\varphi_1 = \omega_1 t$, $\varphi_2 = \omega_2 t + \lambda$, $\omega_2 / \omega_1 = 2n$ (n is a positive integer). Then

$$\bar{F}_1 = \gamma \frac{2\pi}{K} \frac{\kappa}{\sqrt{\kappa^2 + \Lambda^2}} \sin \delta \frac{q^n [(1+q^{2n}) \operatorname{sh} \sigma - (1-q^{2n}) \operatorname{ch} \sigma] \cos \lambda}{1 - 2q^{2n} \operatorname{ch} 2\sigma + q^{4n}}$$

Since $\sin \delta \neq 0$ (otherwise, $|I_3| = I_2$), then Condition 2 of Poincaré's theorem is violated only if

$$(1 - q^{2n}) \operatorname{ch} \sigma - (1 + q^{2n}) \operatorname{sh} \sigma = 0 \tag{2.4}$$

When $n \rightarrow \infty$, according to Lemmas 3 and 4 the function $2F_0 / I_2^2$ tends to $1 / B$, and obviously, $\Lambda \rightarrow C / A (< 1)$. Since Λ is a continuous function of $2F_0 / I_2^2$ in some neighborhood of the point $1 / B$, there exists a number $N_1(A, B, C)$ such that the inequalities

$$0 < \Lambda_1 < \Lambda < \Lambda_2 < 1, \quad \Lambda_i = \text{const}, \quad i = 1, 2$$

are valid for $\omega_2 / \omega_1 = 2n > N_1$. It is obvious that $|\sigma| < \sigma_0$ and $|q| < q_0 < 1$ (as the result $\Lambda_1, \Lambda_2, \sigma_0$ and q_0 depend only on A, B, C). Consequently, there exists a number $N(A, B, C)$ ($N > N_1$) such that equality (2.4) cannot be satisfied for $\omega_2 / \omega_1 = 2n > N$.

Case of dynamic symmetry. In the case $A = B$ it can be shown that in the Deprit variables the Hamiltonian function (1.1) has the form

$$F = \frac{1}{2A} G^2 + \frac{1}{2} \left(\frac{1}{C} - \frac{1}{A} \right) L^2 + P \left[x_0 \left(\frac{H}{G} \sqrt{1 - \left(\frac{L}{G} \right)^2} \sin l + \frac{L}{G} \sqrt{1 - \left(\frac{H}{G} \right)^2} \sin l \cos g + \sqrt{1 - \left(\frac{H}{G} \right)^2} \cos l \sin g \right) + z_0 \left(\frac{LH}{G^2} - \sqrt{1 - \left(\frac{L}{G} \right)^2} \sqrt{1 - \left(\frac{H}{G} \right)^2} \cos g \right) \right]$$

where $(x_0, 0, z_0)$ are the coordinates of the center of gravity in the principal inertia axes, and P is the body's weight. We note that these variables are the variables action-angle of the integrable Euler-Poinsot problem in the symmetric case.

Theorem 4. Let $x_0 \neq 0$ and $A = B > 2C$. Then, for small μ pairs of isolated periodic solutions of the perturbed system are generated on the two-dimensional invariant tori

$$\frac{G}{A} = \pm \left(\frac{1}{C} - \frac{1}{A} \right) L, \quad G \neq 0, \quad G \neq |H|$$

of the reduced Euler-Poinsot problem. They depend analytically on μ and one of the solutions in each pair is stable in the first approximation, while the other is unstable.

This assertion is proved in the same way as Theorem 3.

Note. With the aid of the construction proposed in [7] we can prove that not all the periodic solutions lying on any invariant torus of the Euler-Poinsot problem with a ratio-

nal number of rotations vanish under the addition of perturbation, but at least two are left when μ is small. However we do not know whether they are isolated and depend analytically on μ .

3. The nonexistence of an additional analytic integral of the equations of motion of a nonsymmetric solid body. The generation of a large number of nonsingular periodic solutions of the equations of motion of a nonsymmetric body is incompatible with the integrability of this problem. We can prove the following assertion using Theorem 3.

Theorem 5. The canonic equations of motion of a nonsymmetric heavy solid body with Hamiltonian function (1, 1) do not have a third analytic integral, analytically depending on parameter μ , independent of the classical ones, and being in involution with the area integral.

Proof. Assume the contrary, i. e. let such an integral exist. Then, by virtue of the assumption on involution, there exists an additional independent integral of the reduced canonic system of equations; we denote it by $\Phi = \Phi_0 + \mu \Phi_1 + \dots$. It can be shown that the functions F_0 and Φ_0 are related at all points of the two-dimensional invariant tori of the Euler-Poinsot problem, which we dealt with in Theorem 3 (we call them resonance tori).

In fact, the periodic solutions $\Gamma(\mu)$, arising from the structure of the periodic solutions located on an arbitrary resonance torus T_0^2 of the Euler-Poinsot problem, are not singular. Therefore, as was proved in [6], the functions F and Φ are related at all points of $\Gamma(\mu)$. Let μ tend to zero. The periodic solution $\Gamma(\mu)$ goes into the periodic solution $\Gamma(0)$ of the unperturbed problem, lying on T_0^2 , while the functions F and Φ go into F_0 and Φ_0 , respectively. By continuity the functions F_0 and Φ_0 are related at all points of the trajectory of the periodic solution $\Gamma(0)$. In some neighborhood of torus T_0^2 on which $\Gamma(0)$ lies, we introduce the variables action-angle of the Euler-Poinsot problem, i. e. $(I_1 I_2 \varphi_1 \varphi_2)$. Then F_0 and Φ_0 depend only on I_1 and I_2 (by virtue of the nonsingularity of the reduced Euler-Poinsot problem). Since the functions F_0 and Φ_0 are related on $\Gamma(0)$, the Jacobi matrix $\partial(F_0, \Phi_0) / \partial(I, \varphi)$ is of unity rank when $(I, \varphi) \in \Gamma(0)$. In particular, $\partial(F_0, \Phi_0) / \partial(I_1, I_2) = 0$ at these points. However, the initial Jacobi matrix does not depend on φ , hence its rank equals unity at all points of torus T_0^2 and, consequently, the functions F_0 and Φ_0 are related on the torus. Thus, we have proved that the functions F_0 and Φ_0 are related on the set of all resonance tori of the reduced Euler-Poinsot problem.

It is not difficult to show that the set of all resonance tori possess the following key property: if the analytic function f vanishes on the set, then $f \equiv 0$ in the whole phase space. Since the functions F_0 and Φ_0 are analytic, they are related in the whole phase space, i. e. are simply functionally related. At the same time it can be shown that if an additional independent integral Φ exists, then there exists an analytic integral Φ' such that the functions F_0 and Φ_0' are unrelated (see [6], Sect. 81). The contradiction obtained proves the validity of the theorem.

Note. The basic idea of the arguments advanced is contained in Poincaré's first proof of the general theorem on the nonintegrability of near-integrable canonic equations ([8], Sect. 22). But, as Poincaré himself remarked, his general theorem is inapplicable to the problem being examined ([6], Sect. 86). The success of the proof presented

above consists in the use of the key property of the set of resonance tori of the unperturbed problem.

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REFERENCES

1. Deprit, A., Study of the free rotation of a solid body around a fixed point, using the phase plane, *Mekhanika* (collection of translations), № 2, 1968.
2. Siegel, K. L., Lectures on Celestial Mechanics, Moscow, Izd. Inostr. Lit., 1959.
3. Golubev, V. V., Lectures on the Integration of the Equations of Motion of a Heavy Solid Body Around a Fixed Point, Moscow, Gostekhizdat, 1953.
4. Sadov, Iu. A., The action-angle variables in the Euler-Poinsot problem, *PMM* Vol. 34, № 5, 1970.
5. Arnol'd, V. I., Proof of A. N. Kolmogorov's theorem on the preservation of conditionally-periodic motions under a small variation of the Hamiltonian function, *Uspekhi Matem. Nauk*, Vol. 18, № 5, 1963.
6. Poincaré, H., New Methods in Celestial Mechanics, Collected Works, Vol. 1, Moscow, "Nauka", 1971.
7. Arnol'd, V. I. and Avez, A., Problèmes ergodiques de la mécanique classique, Paris, Gauthier-Villars, 1967.
8. Poincaré, H., On the three-body problem and on the equations of dynamics, Collected Works, Vol. 2, Moscow, "Nauka", 1972.

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ON THE STABILITY OF MOTION IN CRITICAL CASES

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We examine a class of functions of higher than first order in smallness in the equations of perturbed motion, for which the stability problem in critical cases is completely solved in a linear approximation. More precisely, we give a generalization of Malkin's theorem on the singular case of several zero roots to the case when the characteristic equation has pure imaginary roots. We consider the instability question.

1. Let us consider a system of differential equations of perturbed motion (1.1), where the functions Y_i and X_s satisfy conditions (1.2), (1.3)

$$y_i' = q_{i1}x_1 + \dots + q_{in}x_n + Y_i(t, \mathbf{x}, \mathbf{y}), \quad i = 1, \dots, k \quad (1.1)$$

$$x_s' = p_{s1}x_1 + \dots + p_{sn}x_n + X_s(t, \mathbf{x}, \mathbf{y}), \quad s = 1, \dots, n$$

$$Y_i(t, \mathbf{0}, \mathbf{y}) \equiv X_s(t, \mathbf{0}, \mathbf{y}) \equiv 0 \quad (1.2)$$

$$\frac{\|Y(t, \mathbf{x}, \mathbf{y})\| + \|X(t, \mathbf{x}, \mathbf{y})\|}{\|\mathbf{x}\|} \xrightarrow[t \geq 0]{} 0 \quad \text{as } \|\mathbf{x}\| + \|\mathbf{y}\| \rightarrow 0 \quad (1.3)$$